

LYAPUNOV EXPONENTS OF DISCONTINUOUS RANDOM DYNAMICAL SYSTEMS*

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ABSTRACT. Motivated by studying stochastic systems with non-Gaussian Lévy noise, spectral properties for linear discontinuous cocycles are considered. These linear cocycles have countable jump discontinuities in time. Specially, Lyapunov exponents are studied for linear stochastic differential equations with Lévy motions.

1. INTRODUCTION

Lyapunov exponents play an important role in the study of the behavior of dynamical systems. They describe exponential stability of orbits in linear or linearized systems.

For linear cocycles that are continuous in time variable t , existence and properties of Lyapunov exponents are summarized in [2]. These continuous linear cocycles often come from the solution mappings of linear stochastic differential equations (SDE) with (Gaussian) Brownian motions [7, 20, 21]. Lyapunov exponents for linear continuous cocycles in infinite dimensional space have recently been studied [14] (and references therein). Lyapunov exponents for linear stochastic functional differential equations [16, 17, 18] and for linear systems with Poisson noise [13] have also been investigated.

However, as far as we know, Lyapunov exponents for cocycles which are discontinuous (right-continuous with left limits or left-continuous with right limits) in t are not available in literature. Although Mohammed and Scheutzwow [17] studied Lyapunov exponents for linear stochastic functional differential equations driven by semimartingales for $t \in \mathbb{R}_+$, they required that the martingale parts of these semimartingales are continuous in t . Besides, in [15], Mao and Rodkina proved exponential stability of stochastic differential equations driven by discontinuous semimartingales for $t \in \mathbb{R}_+$, although their conditions are stringent or otherwise difficult to verify.

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In the present paper, we study Lyapunov exponents for discontinuous cocycles with two-sided time \mathbb{R} . These cocycles have countable jump discontinuities in time. In fact, we not only generalize the result in [2], but also remove the continuity assumption in [17].

It is worth mentioning that a major source of discontinuous cocycles are solution mappings, after a perfection procedure [25, 11], of stochastic differential equations with Lévy processes. To consider these cocycles with two-sided time \mathbb{R} , we define Lévy processes for two-sided time (i.e., we give a definition of the Lévy processes for $t \leq 0$). Protter [22] did this by using shift operators. We use a different definition appropriate for studying Lyapunov exponents (See Section 2.3.2 for details).

This paper is arranged as follows. In Section 2, we introduce discontinuous cocycles and Lévy processes for $t \geq 0$ and $t \leq 0$, respectively. In Section 3, we consider Lyapunov exponents for linear stochastic differential systems with Lévy processes. In particular, we consider Lyapunov exponents for a linear stochastic differential equation with a α -stable Lévy motion in Section 4. For readers's convenience, a multiplicative ergodic theorem (Theorem 6.2) suitable for linear cocycles which are discontinuous in t is presented in the Appendix.

The following convention will be used throughout the paper: C with or without indices will denote different positive constants (depending on the indices) whose values vary.

2. PRELIMINARIES

In this section, we recall basic concepts and facts that will be needed throughout the paper.

In the following, $|\cdot|$ stands for the length of a vector in \mathbb{R}^d , $\|\cdot\|$ denotes the Hilbert-Schmidt norm of a matrix or the norm of a linear operator and $\langle \cdot, \cdot \rangle$ is the usual scalar product in \mathbb{R}^d .

2.1. Probability space. Let $D^*(\mathbb{R}, \mathbb{R}^d)$ be the set of all functions which are càdlàg (right-continuous with left limits) for $t \geq 0$ and càglàd (left-continuous with right limits) for $t \leq 0$, and which take values in \mathbb{R}^d . We take $\Omega = D^*(\mathbb{R}, \mathbb{R}^d)$, which will be the canonical sample space for stochastic differential equations with two-sided Lévy motions. It can be made a complete and separable metric space when endowed with the Skorohod metric ρ as in [8]: for $x, y \in \Omega$,

$$\rho(x, y) := \inf_{\lambda \in \Lambda} \left\{ \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| + \sum_{m=1}^{\infty} \frac{1}{2^m} \min \left\{ 1, \rho_m^\circ(x^m, y^m) \right\} \right\},$$

where $x^m(t) := g_m(t)x(t)$ and $y^m(t) := g_m(t)y(t)$ with

$$g_m(t) := \begin{cases} 1, & \text{if } |t| \leq m, \\ m+1-|t|, & \text{if } m < |t| < m+1, \\ 0, & \text{if } |t| \geq m+1, \end{cases}$$

and

$$\rho_m^\circ(x, y) := \sup_{|t| \leq m} |x(t) - y(\lambda(t))|.$$

Here Λ denotes the set of strictly increasing and continuous functions from \mathbb{R} to \mathbb{R} . We identify a function $\omega(t)$ with a (canonical) sample ω in the sample space Ω .

The Borel σ -field in the sample space Ω , under the topology induced by the Skorohod metric ρ , is denoted by \mathcal{F} . Note that $\mathcal{F} = \sigma(\omega(t), t \in \mathbb{R})$, as known in [8]. Let \mathbb{P} be the unique probability measure which makes the canonical process a Lévy process for $t \in \mathbb{R}$ (Definition 2.2 and 2.4). And we have the complete natural filtration $\mathcal{F}_s^t := \sigma(\omega(u) : s \leq u \leq t) \vee \mathcal{N}$ for $s \leq t$ with respect to \mathbb{P} . Here \mathcal{N} is the set of all null events under \mathbb{P} .

2.2. Definition of discontinuous cocycles and linear cocycles. Define for each $t \in \mathbb{R}$

$$(\theta_t \omega)(\cdot) = \omega(t + \cdot) - \omega(t), \quad \omega \in \Omega.$$

Then $\{\theta_t\}$ is a one-parameter group (or a *flow*, or a *deterministic dynamical system*) on Ω . In fact, Ω is invariant with respect to $\{\theta_t\}$, i.e.

$$\theta_t^{-1} \Omega = \Omega, \quad \text{for all } t \in \mathbb{R},$$

and \mathbb{P} is $\{\theta_t\}$ -invariant, i.e.

$$\mathbb{P}(\theta_t^{-1}(B)) = \mathbb{P}(B), \quad \text{for all } B \in \mathcal{F}, t \in \mathbb{R}.$$

Thus $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system (DS), or also called a driving dynamical system. The metric DS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is called ergodic, if all measurable $\{\theta_t\}$ -invariant sets have probability 0 or 1. (see [2])

Definition 2.1. Let $(\mathbb{X}, \mathcal{B})$ be a measurable space. For a mapping

$$\begin{aligned} \varphi : \mathbb{R} \times \Omega \times \mathbb{X} &\mapsto \mathbb{X}, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x), \\ \varphi(t, \omega) &:= \varphi(t, \omega, \cdot) : \mathbb{X} \mapsto \mathbb{X}, \end{aligned}$$

(i) *Discontinuous cocycle property:* $\varphi(t, \omega)$ forms a (perfect) discontinuous cocycle over θ if it is càdlàg for $t \geq 0$ and càglàd for $t \leq 0$, and further satisfies the following conditions

$$\varphi(0, \omega) = id_{\mathbb{X}}, \tag{1}$$

$$\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega), \tag{2}$$

for all $s, t \in \mathbb{R}$ and $\omega \in \Omega$; $\varphi(t, \omega)$ is called a crude discontinuous cocycle over θ if it is càdlàg for $t \geq 0$ and càglàd for $t \leq 0$ and (1) holds, and for every $s \in \mathbb{R}$ there exists a \mathbb{P} -null set $N_s \subset \Omega$ such that (2) holds for all $t \in \mathbb{R}$ and $\omega \in \Omega \setminus N_s$.

(ii) *Linearity:* $\varphi(t, \omega)$ is called a linear cocycle if for each $t \in \mathbb{R}$ and $\omega \in \Omega$, $\varphi(t, \omega)$ is a linear operator in \mathbb{X} .

2.3. Lévy processes. We now define two-sided Lévy processes (also called Lévy motions).

2.3.1. Lévy processes for $t \geq 0$.

Definition 2.2. A process $L = (L_t)_{t \geq 0}$ with $L_0 = 0$, a.s., is a d -dimensional Lévy process for $t \geq 0$ if

(i) L has independent increments; that is, $L_t - L_s$ is independent of $L_v - L_u$ if $(u, v) \cap (s, t) = \emptyset$;

(ii) L has stationary increments; that is, $L_t - L_s$ has the same distribution as $L_v - L_u$ if $t - s = v - u > 0$;

(iii) L_t is right continuous with left limit.

The characteristic function of L_t is given by

$$\mathbb{E}(\exp\{i\langle z, L_t \rangle\}) = \exp\{t\Psi(z)\}, \quad z \in \mathbb{R}^d.$$

The function $\Psi : \mathbb{R}^d \rightarrow \mathbb{C}$ is called the characteristic exponent of the Lévy process L . By the Lévy-Khintchine formula, there exist a nonnegative-definite $d \times d$ matrix Q , a vector $\gamma \in \mathbb{R}^d$, and a measure ν on \mathbb{R}^d satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|u|^2 \wedge 1) \nu(du) < \infty, \quad (3)$$

such that

$$\begin{aligned} \Psi(z) &= -\frac{1}{2}\langle z, Qz \rangle + i\langle z, \gamma \rangle \\ &\quad + \int_{\mathbb{R}^d} (e^{i\langle z, u \rangle} - 1 - i\langle z, u \rangle 1_{|u| \leq \delta}) \nu(du), \end{aligned} \quad (4)$$

where $\delta > 0$ is a constant. Here ν is called a Lévy measure or a Lévy jump measure.

Set $\kappa_t := L_t - L_{t-}$. Then κ defines a stationary $(\mathcal{F}_0^t)_{t \geq 0}$ -adapted Poisson point process with values in $\mathbb{R}^d \setminus \{0\}$ and characteristic measure ν (c.f. [10]). Let $N_\kappa((0, t], du)$ be the counting measure of κ_t , i.e., for $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$

$$N_\kappa((0, t], B) := \#\{0 < s \leq t : \kappa_s \in B\},$$

where $\#$ denotes the cardinality of a set. The compensator measure of N_κ is given by

$$\tilde{N}_\kappa((0, t], du) := N_\kappa((0, t], du) - t\nu(du).$$

The Lévy-Itô theorem states that there exist a vector $b \in \mathbb{R}^d$, a d' -dimensional $(\mathcal{F}_0^t)_{t \geq 0}$ -Brownian motion W_t , with $0 \leq d' \leq d$, and a $d \times d'$ matrix A , such that L can be represented as

$$\begin{aligned} L_t &= bt + AW_t + \int_0^t \int_{|u| \leq \delta} u \tilde{N}_\kappa(ds, du) \\ &\quad + \int_0^t \int_{|u| > \delta} u N_\kappa(ds, du). \end{aligned} \quad (5)$$

We now introduce a special class of Lévy motions, i.e., α -stable Lévy processes L_t^α .

Definition 2.3. *A d -dimensional Lévy process L is called a stable process with index $\alpha \in (0, 2]$ if its characteristic exponent Ψ has the following special form:*

$$\Psi(kz) = k^\alpha \Psi(z),$$

for every $k > 0$ and every $z \in \mathbb{R}^d$.

In fact, for a d -dimensional α -stable Lévy motion L_t^α , its Lévy-Itô representation is given by

$$L_t^\alpha = bt + \int_0^t \int_{|u| \leq \delta} u \tilde{N}_\kappa(ds, du) + \int_0^t \int_{|u| > \delta} u N_\kappa(ds, du).$$

2.3.2. Lévy processes for $t \leq 0$.

Definition 2.4. A process $L^- = (L_t^-)_{t \leq 0}$ with $L_0^- = 0$, a.s., is a d -dimensional Lévy process for $t \leq 0$ if

- (i) L^- has independent increments; that is, $L_t^- - L_s^-$ is independent of $L_v^- - L_u^-$ if $(v, u) \cap (t, s) = \emptyset$;
- (ii) L^- has stationary increments; that is, $L_t^- - L_s^-$ has the same distribution as $L_v^- - L_u^-$ if $t - s = v - u < 0$;
- (iii) L_t^- is left continuous with right limit.

The corresponding characteristic function, characteristic exponent and the Lévy-Khintchine formula are the same as those for the Lévy process $L = (L_t)_{t \geq 0}$.

Set $\kappa_t^- := L_t^- - L_{t+}^-$. Then κ^- defines a stationary $(\mathcal{F}_t^0)_{t \leq 0}$ -adapted Poisson point process with values in $\mathbb{R}^d \setminus \{0\}$ and characteristic measure ν^- (c.f.[10]). Define

$$N_{\kappa^-}([t, 0), B) := \#\{t \leq s < 0 : \kappa_s^- \in B\},$$

for $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. The compensator measure of N_{κ^-} is given by

$$\tilde{N}_{\kappa^-}([t, 0), du) := N_{\kappa^-}([t, 0), du) + t\nu^-(du),$$

where ν^- is the Lévy measure for L^- . By the similar proof as that of the Lévy-Itô theorem in [24], we obtain that there exist a vector $b^- \in \mathbb{R}^d$, a d'' -dimensional $(\mathcal{F}_t^0)_{t \leq 0}$ -Brownian motion W_t^- , with $0 \leq d'' \leq d$, and a $d \times d''$ matrix A^- , such that L^- can be represented as

$$\begin{aligned} L_t^- &= -b^-t - A^-W_t^- - \int_t^0 \int_{|u| \leq \delta} u \tilde{N}_{\kappa^-}(ds, du) \\ &\quad - \int_t^0 \int_{|u| > \delta} u N_{\kappa^-}(ds, du). \end{aligned} \quad (6)$$

We can thus define d -dimensional α -stable Lévy motion $L_t^{\alpha-}$, and its Lévy-Itô representation is now

$$L_t^{\alpha-} = -b^-t - \int_t^0 \int_{|u| \leq \delta} u \tilde{N}_{\kappa^-}(ds, du) - \int_t^0 \int_{|u| > \delta} u N_{\kappa^-}(ds, du).$$

3. LYAPUNOV EXPONENTS OF LINEAR SDES WITH LÉVY MOTIONS

Consider a linear stochastic system in \mathbb{R}^2 with Lévy motions:

$$\begin{cases} dX_t^1 = 2X_t^1 dt + X_t^1 dL_t^1, & X_0^1 = x^1, & t \geq 0, \\ dX_t^2 = -4X_t^2 dt + X_t^2 dL_t^2, & X_0^2 = x^2, & t \geq 0, \end{cases} \quad (7)$$

where L_t^1 and L_t^2 are two one-dimensional independent Lévy processes with the same Lévy measure ν and the following Lévy-Itô representations,

$$L_t^1 = \int_0^t \int_{|u| \leq \delta} u \tilde{N}_{\kappa^1}(ds, du), \quad L_t^2 = \int_0^t \int_{|u| \leq \delta} u \tilde{N}_{\kappa^2}(ds, du),$$

where $0 < \delta < 1$ is a constant, $\kappa_t^1 := L_t^1 - L_{t-}^1$ and $\kappa_t^2 := L_t^2 - L_{t-}^2$. By the Itô formula, we obtain the solution of Eq.(7)

$$\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} M_t^1 & 0 \\ 0 & M_t^2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix},$$

where

$$M_t^1 = \exp \left\{ \left[2 + \int_{|u| \leq \delta} (\log(1+u) - u) \nu(du) \right] t + \int_0^t \int_{|u| \leq \delta} \log(1+u) \tilde{N}_{\kappa^1}(ds, du) \right\},$$

$$M_t^2 = \exp \left\{ \left[-4 + \int_{|u| \leq \delta} (\log(1+u) - u) \nu(du) \right] t + \int_0^t \int_{|u| \leq \delta} \log(1+u) \tilde{N}_{\kappa^2}(ds, du) \right\}.$$

Define a càdlàg cocycle

$$\varphi(t, \omega) = \begin{pmatrix} M_t^1 & 0 \\ 0 & M_t^2 \end{pmatrix}.$$

Firstly, we justify that $\varphi(t, \omega)$ satisfies the integrability condition (20). Rewrite Eq.(7) as

$$\begin{cases} dX_t = aX_t dt + \sigma^1 X_t dL_t^1 + \sigma^2 X_t dL_t^2, \\ X_0 = x, \end{cases}$$

where

$$X_t = \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix}, \quad x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix},$$

$$a = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Applying the Itô formula to $\log |X_t|$, we infer that

$$\begin{aligned} \log |X_t| &= \log |X_0| + \int_0^t \frac{X_s^i a_{ik} X_s^k}{|X_s|^2} ds \\ &+ \int_0^t \int_{|u| \leq \delta} [\log |X_{s-} + u \sigma^j X_{s-}| - \log |X_{s-}|] \tilde{N}_{\kappa^j}(ds, du) \\ &+ \sum_{j=1}^2 \int_0^t \int_{|u| \leq \delta} \left[\log |X_{s-} + u \sigma^j X_{s-}| - \log |X_{s-}| - u \frac{X_{s-}^i \sigma_{ik}^j X_{s-}^k}{|X_{s-}|^2} \right] \nu(du) ds. \end{aligned}$$

So,

$$\sup_{0 \leq t \leq 1} \log^+ \|\varphi(t, \omega)\| \leq \sup_{|x|=1} \sup_{0 \leq t \leq 1} |\log |X_t|| \leq 4 + I_1 + I_2,$$

where

$$I_1 = \sup_{|x|=1} \sup_{0 \leq t \leq 1} \left| \int_0^t \int_{|u| \leq \delta} [\log |X_{s-} + u \sigma^j X_{s-}| - \log |X_{s-}|] \tilde{N}_{\kappa^j}(ds, du) \right|,$$

$$I_2 = \sup_{|x|=1} \sup_{0 \leq t \leq 1} \left| \sum_{j=1}^2 \int_0^t \int_{|u| \leq \delta} \left[\log |X_{s-} + u \sigma^j X_{s-}| - \log |X_{s-}| - u \frac{X_{s-}^i \sigma_{ik}^j X_{s-}^k}{|X_{s-}|^2} \right] \nu(du) ds \right|.$$

For I_1 , by BDG inequality, mean value theorem and Hölder's inequality, we have

$$\mathbb{E} I_1 \leq \mathbb{E} \left[\sup_{|x|=1} \int_0^1 \int_{|u| \leq \delta} (\log |X_{s-} + u \sigma^j X_{s-}| - \log |X_{s-}|)^2 N_{\kappa^j}(ds, du) \right]^{\frac{1}{2}}$$

$$\begin{aligned}
&\leq \mathbb{E} \left[\sup_{|x|=1} \int_0^1 \int_{|u| \leq \delta} \frac{|u|^2}{(1-|u|)^2} N_{\kappa^j}(\mathrm{d}s, \mathrm{d}u) \right]^{\frac{1}{2}} \\
&\leq \left[\mathbb{E} \left(\int_0^1 \int_{|u| \leq \delta} \frac{|u|^2}{(1-\delta)^2} N_{\kappa^j}(\mathrm{d}s, \mathrm{d}u) \right) \right]^{\frac{1}{2}} \\
&= \frac{1}{1-\delta} \left[\int_{|u| \leq \delta} |u|^2 \nu(\mathrm{d}u) \right]^{\frac{1}{2}}.
\end{aligned}$$

For I_2 , by mean value theorem, it holds that

$$\begin{aligned}
\mathbb{E} I_2 &\leq \mathbb{E} \left[\sup_{|x|=1} \sup_{0 \leq t \leq 1} \sum_{j=1}^2 \int_0^t \int_{|u| \leq \delta} \left| \log |X_{s-} + u \sigma^j X_{s-}| - \log |X_{s-}| \right. \right. \\
&\quad \left. \left. - u \frac{X_{s-}^i \sigma_{ik}^j X_{s-}^k}{|X_{s-}|^2} \right| \nu(\mathrm{d}u) \mathrm{d}s \right] \\
&\leq \mathbb{E} \left[\sup_{|x|=1} \sup_{0 \leq t \leq 1} 4 \int_0^t \int_{|u| \leq \delta} \frac{|u|^2}{(1-|u|)^2} \nu(\mathrm{d}u) \mathrm{d}s \right] \\
&\leq \frac{4}{(1-\delta)^2} \int_{|u| \leq \delta} |u|^2 \nu(\mathrm{d}u).
\end{aligned}$$

Thus,

$$\mathbb{E} \alpha^+ = \mathbb{E} \left(\sup_{0 \leq t \leq 1} \log^+ \|\varphi(t, \omega)\| \right) < \infty.$$

As in linear algebra, we find the inverse

$$\varphi(t, \omega)^{-1} = \begin{pmatrix} (M_t^1)^{-1} & 0 \\ 0 & (M_t^2)^{-1} \end{pmatrix}.$$

Simple calculations lead to

$$\begin{aligned}
\log \|\varphi(t, \omega)^{-1}\| &= \frac{1}{2} \log ((M_t^1)^{-2} + (M_t^2)^{-2}) \\
&= \frac{1}{2} \log ((M_t^1)^2 + (M_t^2)^2) - \log M_t^1 - \log M_t^2 \\
&= \log \|\varphi(t, \omega)\| - \log M_t^1 - \log M_t^2 \\
&\leq \log \|\varphi(t, \omega)\| + |\log M_t^1| + |\log M_t^2|.
\end{aligned}$$

By Jensen's inequality, we obtain

$$\begin{aligned}
\mathbb{E} \left(\sup_{0 \leq t \leq 1} \log^+ \|\varphi(t, \omega)^{-1}\| \right) &\leq \mathbb{E} \left(\sup_{0 \leq t \leq 1} \log^+ \|\varphi(t, \omega)\| \right) + \mathbb{E} \left(\sup_{0 \leq t \leq 1} |\log M_t^1| \right) \\
&\quad + \mathbb{E} \left(\sup_{0 \leq t \leq 1} |\log M_t^2| \right) \\
&= \mathbb{E} \alpha^+ + \mathbb{E} \left(\sup_{0 \leq t \leq 1} |\log M_t^1| \right) + \mathbb{E} \left(\sup_{0 \leq t \leq 1} |\log M_t^2| \right).
\end{aligned}$$

For the second term in the right hand side of the above inequality, it follows from BDG inequality and the Hölder inequality that

$$\begin{aligned}
\mathbb{E} \left(\sup_{0 \leq t \leq 1} |\log M_t^1| \right) &\leq \mathbb{E} \left(\sup_{0 \leq t \leq 1} \left| 2 + \int_{|u| \leq \delta} (\log(1+u) - u) \nu(du) \right| t \right) \\
&\quad + \mathbb{E} \left(\sup_{0 \leq t \leq 1} \left| \int_0^t \int_{|u| \leq \delta} \log(1+u) \tilde{N}_{\kappa^1}(ds, du) \right| \right) \\
&\leq \left| 2 + \int_{|u| \leq \delta} (\log(1+u) - u) \nu(du) \right| \\
&\quad + \mathbb{E} \left(\int_0^1 \int_{|u| \leq \delta} (\log(1+u))^2 N_{\kappa^1}(ds, du) \right)^{\frac{1}{2}} \\
&\leq 2 + \int_{|u| \leq \delta} |\log(1+u) - u| \nu(du) \\
&\quad + \left(\mathbb{E} \left(\int_0^1 \int_{|u| \leq \delta} (\log(1+u))^2 N_{\kappa^1}(ds, du) \right) \right)^{\frac{1}{2}} \\
&= 2 + \int_{|u| \leq \delta} |\log(1+u) - u| \nu(du) \\
&\quad + \left(\int_{|u| \leq \delta} (\log(1+u))^2 \nu(du) \right)^{\frac{1}{2}}.
\end{aligned}$$

Since $|\log(1+u) - u| \leq C|u|^2$ and $\log^2(1+u) \leq C|u|^2$ for $|u| \leq \delta$, we thus have by (3)

$$\mathbb{E} \left(\sup_{0 \leq t \leq 1} |\log M_t^1| \right) < \infty.$$

Similarly, we also obtain $\mathbb{E} \left(\sup_{0 \leq t \leq 1} |\log M_t^2| \right) < \infty$. Thus,

$$\mathbb{E} \alpha^- = \mathbb{E} \left(\sup_{0 \leq t \leq 1} \log^+ \|\varphi(t, \omega)^{-1}\| \right) < \infty.$$

By the strong law of large numbers, it follows that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_{|u| \leq \delta} \log(1+u) \tilde{N}_{\kappa^i}(ds, du) = 0.$$

Thus, by Theorem 6.2 we have

$$\lim_{t \rightarrow \infty} (\varphi(t, \omega)^* \varphi(t, \omega))^{1/2t} = \lim_{t \rightarrow \infty} \left(\begin{pmatrix} (M_t^1)^2 & 0 \\ 0 & (M_t^2)^2 \end{pmatrix} \right)^{1/2t} = \begin{pmatrix} M^1 & 0 \\ 0 & M^2 \end{pmatrix} =: \Phi(\omega),$$

where

$$\begin{aligned}
M^1 &= \exp \left\{ 2 + \int_{|u| \leq \delta} (\log(1+u) - u) \nu(du) \right\}, \\
M^2 &= \exp \left\{ -4 + \int_{|u| \leq \delta} (\log(1+u) - u) \nu(du) \right\}.
\end{aligned}$$

By direct calculations, we get eigenvalues $e^{\lambda_2} < e^{\lambda_1}$ of $\Phi(\omega)$, where

$$\begin{aligned}\lambda_2 &= -4 + \int_{|u| \leq \delta} (\log(1+u) - u) \nu(du), \\ \lambda_1 &= 2 + \int_{|u| \leq \delta} (\log(1+u) - u) \nu(du),\end{aligned}$$

and corresponding eigenspaces $U_2 = \{(x^1, x^2) \in \mathbb{R}^2 | x^1 = 0\}$ and $U_1 = \{(x^1, x^2) \in \mathbb{R}^2 | x^2 = 0\}$.

Take $V_3 = \{0\}$, $V_2 = U_2$ and $V_1 = \mathbb{R}^2 \setminus \{0\}$,

$$\begin{aligned}\lambda(\omega, x) = \lambda_2 &\Leftrightarrow x \in U_2 = V_2 \setminus V_3, \\ \lambda(\omega, x) = \lambda_1 &\Leftrightarrow x \in U_1 = V_1 \setminus V_2.\end{aligned}$$

We reexamine the MET for this system but with two-sided Lévy motions L_t^1 and L_t^2 . Define Lévy processes L_t^{1-}, L_t^{2-} for $t \leq 0$ such that

$$L_t^{1-} = - \int_t^0 \int_{|u| \leq \delta} u \tilde{N}_{\kappa^{1-}}(ds, du), \quad L_t^{2-} = - \int_t^0 \int_{|u| \leq \delta} u \tilde{N}_{\kappa^{2-}}(ds, du),$$

where $\kappa^{1-} = L_t^{1-} - L_{t+}^{1-}$ and $\kappa^{2-} = L_t^{2-} - L_{t+}^{2-}$, L_t^{i-} is independent of L_t^j for $j = 1, 2$ and L_t^{k-} for $k = 1, 2, k \neq i$. Set

$$\hat{L}_t^1 = \begin{cases} L_t^1, & t \geq 0, \\ L_t^{1-}, & t \leq 0, \end{cases} \quad \hat{L}_t^2 = \begin{cases} L_t^2, & t \geq 0, \\ L_t^{2-}, & t \leq 0. \end{cases}$$

Now consider the following system

$$\begin{cases} dX_t^1 = 2X_t^1 dt + X_t^1 d\hat{L}_t^1, & X_0^1 = x^1, \\ dX_t^2 = -4X_t^2 dt + X_t^2 d\hat{L}_t^2, & X_0^2 = x^2, \end{cases} \quad (8)$$

where the corresponding stochastic integrals are understood as the forward Itô integrals for $t \geq 0$ ([10]), and the backward Itô integrals for $t \leq 0$ ([5]). By the similar deduction as that for $t \geq 0$ in the first part of this example, we conclude that Eq.(8) generates a linear discontinuous cocycle $\varphi(t, \omega)$ for $t \in \mathbb{R}$ and $\omega \in \Omega$. By Theorem 6.2,

$$\begin{aligned}E_1 &= V_1 \cap V_{2+1-1}^- = V_1 \cap V_2^- = V_2^- = U_1, \\ E_2 &= V_2 \cap V_{2+1-2}^- = V_2 \cap V_1^- = U_2,\end{aligned}$$

and

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log |\varphi(t, \omega)x| = \lambda_i(\omega) \iff x \in E_i(\omega) \setminus \{0\}, \quad i = 1, 2.$$

4. LYAPUNOV EXPONENTS OF LINEAR SDEs WITH α -STABLE LÉVY MOTIONS

Example 4.1. Consider the following linear SDE

$$\begin{cases} dY_t = Y_t d\Gamma_t, & t > 0, \\ Y_0 = y, \end{cases}$$

where Γ_t is a one-dimensional Lévy process. By [23, Theorem 37, p.84], its solution is

$$Y_t(y) = y \exp \left\{ \Gamma_t - \frac{1}{2} [\Gamma, \Gamma]_t^c \right\} \prod_{0 < s \leq t} (1 + \Delta \Gamma_s) \exp \{-\Delta \Gamma_s\},$$

where $[\Gamma, \Gamma]_t^c$ is the quadratic variation for the continuous local martingale part of Γ_t . Set $\varphi(t, \omega, y) := Y_t(y)$ and in fact, $\varphi(t, \omega, y)$ is a linear càdlàg cocycle. However, integrability of (20) seems difficult to be verified, via the representation of $\varphi(t, \omega, y)$.

But if we consider more specific lévy processes, we may be able to verify the integrability of (20) and calculate Lyapunov exponents. To this end, we study linear SDEs with α -stable Lévy motions.

Consider the following d -dimensional linear SDE

$$\begin{cases} dX_t = aX_t dt + \sigma_i X_t dL_t^i, & t > 0, \\ X_0 = x, \end{cases} \quad (9)$$

where a, σ_i are $d \times d$ real matrices, and L_t^i , for $1 \leq i \leq q$, are one-dimensional independent α -stable Lévy motions. By Section 2.3, the above equation can be written as follows

$$\begin{cases} dX_t = (a + \sigma_i b^i) X_t dt + \int_{|u| \leq \delta} \sigma_i X_t u \tilde{N}_{\kappa^i}(dt, du) \\ \quad + \int_{|u| > \delta} \sigma_i X_t u N_{\kappa^i}(dt, du), & t > 0, \\ X_0 = x. \end{cases} \quad (10)$$

To study Eq.(10), we introduce an auxiliary equation

$$\begin{cases} d\psi_t = \int_{|u| \leq \delta} \sigma_i \psi_t u \tilde{N}_{\kappa^i}(dt, du), & t > 0, \\ \psi_0 = I, \end{cases} \quad (11)$$

where I denotes the unit matrix. By [1, Theorem 6.2.3, p.367], Eq.(11) has a unique solution ψ which is $(\mathcal{F}_0^t)_{t \geq 0}$ -adapted and càdlàg. And by [23, Theorem 63, p.342], $\psi_t(\omega)$ is invertible for all $\omega \in \Omega$ and $t \geq 0$. Furthermore, ψ_t^{-1} solves the following equation

$$\begin{cases} d\psi_t^{-1} = -\psi_t^{-1} dZ_t + \psi_t^{-1} d \sum_{0 < s \leq t} (I + \Delta Z_s)^{-1} (\Delta Z_s)^2, & t > 0, \\ \psi_0^{-1} = I, \end{cases} \quad (12)$$

where $Z_t := \int_0^t \int_{|u| \leq \delta} \sigma_i u \tilde{N}_{\kappa^i}(ds, du)$. Moreover, it follows from [1, Corollary 6.4.11, p.391] that ψ and ψ^{-1} are two càdlàg cocycles.

Next, look at a random integral equation

$$\begin{aligned} X_t = & \psi_t \left\{ x + \int_0^t \psi_s^{-1} (a + \sigma_i b^i) X_s ds \right. \\ & \left. + \int_0^t \int_{|u| > \delta} \psi_s^{-1} \sigma_i X_s u N_{\kappa^i}(ds, du) \right\}. \end{aligned} \quad (13)$$

We have the following result.

Theorem 4.2. *Eq.(10) and Eq.(13) are equivalent: Every càdlàg $(\mathcal{F}_0^t)_{t \geq 0}$ -adapted solution of Eq.(10) is a solution of Eq.(13). Conversely, every solution of Eq.(13) has a version which satisfies Eq.(10).*

Proof. We start from Eq.(13).

$$\begin{aligned} dX_t = & \int_{|u| \leq \delta} \sigma_i \psi_t u \tilde{N}_{\kappa^i}(dt, du) \cdot \left\{ x + \int_0^t \psi_s^{-1} (a + \sigma_i b^i) X_s ds \right. \\ & \left. + \int_0^t \int_{|u| > \delta} \psi_s^{-1} \sigma_i X_s u N_{\kappa^i}(ds, du) \right\} \end{aligned}$$

$$\begin{aligned}
& + \psi_t \psi_t^{-1} (a + \sigma_i b^i) X_t dt + \int_{|u| > \delta} \psi_t \psi_t^{-1} \sigma_i X_t u N_{\kappa^i}(dt, du) \\
& = \int_{|u| \leq \delta} \sigma_i \psi_t u \tilde{N}_{\kappa^i}(dt, du) \cdot \psi_t^{-1} X_t + (a + \sigma_i b^i) X_t dt \\
& \quad + \int_{|u| > \delta} \sigma_i X_t u N_{\kappa^i}(dt, du) \\
& = \int_{|u| \leq \delta} \sigma_i X_t u \tilde{N}_{\kappa^i}(dt, du) + (a + \sigma_i b^i) X_t dt \\
& \quad + \int_{|u| > \delta} \sigma_i X_t u N_{\kappa^i}(dt, du).
\end{aligned}$$

From this, we know that every solution of Eq.(13) has a version which satisfies Eq.(10), and every càdlàg $(\mathcal{F}_0^t)_{t \geq 0}$ -adapted solution of Eq.(10) is a solution of Eq.(13). \square

We collect some properties of the solution to Eq.(13) in the following theorem.

Theorem 4.3. *Eq.(13) has a unique càdlàg solution $X_t(x)$. Define $\varphi : \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$ by $\varphi(t, \omega)x := \varphi(t, \omega, x) := X_t(x)$ for $t \in \mathbb{R}_+$, $\omega \in \Omega$ and $x \in \mathbb{R}^d$. Then the following claims hold:*

- (i) *The map $\varphi(t, \omega, \cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is continuous and linear for $t \in \mathbb{R}_+$ and $\omega \in \Omega$.*
- (ii) *φ is a crude cocycle, i.e. for $s \in \mathbb{R}_+$ there exists a P -null set $N_s \subset \Omega$ such that*

$$\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega),$$

for all $t \in \mathbb{R}_+$ and $\omega \in N_s$.

- (iii) *$\varphi(t, \omega)$ is strongly measurable.*

Proof. Fix $\omega \in \Omega$ and $0 < T < \infty$. Define $X_t^0 := x$ and

$$\begin{aligned}
X_t^n &:= \psi_t \left\{ x + \int_0^t \psi_s^{-1} (a + \sigma_i b^i) X_s^{n-1} ds \right. \\
&\quad \left. + \int_0^t \int_{|u| > \delta} \psi_s^{-1} \sigma_i X_s^{n-1} u N_{\kappa^i}(ds, du) \right\}, \quad n \geq 1.
\end{aligned} \tag{14}$$

Set

$$\begin{aligned}
C_1(\omega) &= \sup_{0 \leq t \leq T} \|\psi_t\|, \quad C_2(\omega) = \sup_{0 \leq t \leq T} \|\psi_t^{-1}\|, \\
C_3(\omega) &= C_1(\omega) C_2(\omega) \|a + \sigma_i b^i\| + C_1(\omega) C_2(\omega), \\
\xi_t(\omega) &= t + \|\sigma_i\| \int_0^t \int_{|u| > \delta} |u| N_{\kappa^i}(ds, du).
\end{aligned}$$

Thus,

$$\begin{aligned}
|X_t^n| &\leq \|\psi_t\| \left\{ |x| + \int_0^t \|\psi_s^{-1}\| \|a + \sigma_i b^i\| |X_s^{n-1}| ds \right. \\
&\quad \left. + \int_0^t \int_{|u| > \delta} \|\psi_s^{-1}\| \|\sigma_i\| |X_s^{n-1}| |u| N_{\kappa^i}(ds, du) \right\} \\
&\leq C_1(\omega) |x| + C_3(\omega) \int_0^t |X_s^{n-1}| d\xi_s,
\end{aligned}$$

which, together with (31) in [17], yields by induction

$$|X_t^{n+1} - X_t^n| \leq \frac{(C_3 \xi_t)^n}{n!} \sup_{0 \leq t \leq T} |X_t^1 - X_t^0|, \quad n = 0, 1, 2, \dots$$

Let \mathbb{F} be the space of all bounded maps $f : [0, T] \times B \mapsto \mathbb{R}^d$, where $B = \{x \in \mathbb{R}^d : |x| \leq 1\}$, such that for each $x \in B$, $f(\cdot, x)$ is càdlàg and for $t \in [0, T]$, $f(t, \cdot)$ is continuous and linear. And \mathbb{F} is a Banach space under the norm

$$\|f\|_{\mathbb{F}} = \sup_{0 \leq t \leq T} \sup_{x \in B} |f(t, x)|.$$

Set

$$X_t := \sum_{n=0}^{\infty} (X_t^{n+1} - X_t^n) + x.$$

So X_t is well defined for $t \in [0, T]$ and belongs to \mathbb{F} . Moreover

$$\begin{aligned} \|X^n - X\|_{\mathbb{F}} &\leq \sum_{k=n}^{\infty} \|X^{k+1} - X^k\|_{\mathbb{F}} \\ &\leq \left(\sum_{k=n}^{\infty} \frac{(C_3 \xi_T)^k}{k!} \right) \|X^1 - X^0\|_{\mathbb{F}}. \end{aligned}$$

Thus $\{X^n\}$ converges to X in \mathbb{F} for $n \rightarrow \infty$. Taking the limit on both sides of (14), we see that X solves Eq.(13). By the similar proof as that for existence, we also have uniqueness of the solution of Eq.(13). Next, X extends by linearity to a map $X(\cdot, \omega, \cdot) : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}^d$ such that for each $t \in \mathbb{R}_+$, $X(t, \omega, \cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is continuous and linear; and for each $x \in \mathbb{R}^d$, $X(\cdot, \omega, x) : \mathbb{R}_+ \mapsto \mathbb{R}^d$ is càdlàg.

To prove (ii), we introduce the following equation

$$\begin{aligned} X_{s,t} = \psi_{s,t} \Big\{ &x + \int_s^t \psi_{s,r}^{-1} (a + \sigma_i b^i) X_{s,r} dr \\ &+ \int_s^t \int_{|u| > \delta} \psi_{s,r}^{-1} \sigma_i X_{s,r} u N_{\kappa^i}(dr, du) \Big\}, \quad 0 \leq s < t, \end{aligned} \quad (15)$$

where $\psi_{s,t}$ and $\psi_{s,t}^{-1}$ solve the following equations

$$\psi_{s,t} = I + \int_s^t \int_{|u| \leq \delta} \sigma_i \psi_{s,r} u \tilde{N}_{\kappa^i}(dr, du)$$

and

$$\psi_{s,t}^{-1} = I - \int_s^t \psi_{s,r}^{-1} dZ_r + \int_s^t \psi_{s,r}^{-1} d \sum_{0 < u \leq r} (I + \Delta Z_u)^{-1} (\Delta Z_u)^2,$$

respectively.

Furthermore,

$$\begin{aligned} X_{0,s+t} = \psi_{0,s+t} \Big\{ &x + \int_0^{s+t} \psi_{0,r}^{-1} (a + \sigma_i b^i) X_{0,r} dr \\ &+ \int_0^{s+t} \int_{|u| > \delta} \psi_{0,r}^{-1} \sigma_i X_{0,r} u N_{\kappa^i}(dr, du) \Big\} \end{aligned}$$

$$\begin{aligned}
&= \psi_{s,s+t}\psi_{0,s}\left\{x + \int_0^s \psi_{0,r}^{-1}(a + \sigma_i b^i)X_{0,r}dr \right. \\
&\quad \left. + \int_0^s \int_{|u|>\delta} \psi_{0,r}^{-1}\sigma_i X_{0,r}uN_{\kappa^i}(dr, du) \right. \\
&\quad \left. + \int_s^{s+t} \psi_{0,r}^{-1}(a + \sigma_i b^i)X_{0,r}dr \right. \\
&\quad \left. + \int_s^{s+t} \int_{|u|>\delta} \psi_{0,r}^{-1}\sigma_i X_{0,r}uN_{\kappa^i}(dr, du) \right\} \\
&= \psi_{s,s+t}\left\{X_{0,s} + \int_s^{s+t} \psi_{s,r}^{-1}(a + \sigma_i b^i)X_{0,r}dr \right. \\
&\quad \left. + \int_s^{s+t} \int_{|u|>\delta} \psi_{s,r}^{-1}\sigma_i X_{0,r}uN_{\kappa^i}(dr, du) \right\},
\end{aligned}$$

where the last step is based on $\psi_{0,s}\psi_{0,r}^{-1} = \psi_{s,r}^{-1}$ for $0 \leq s \leq r$ (c.f. [4]). By uniqueness of the solution for Eq.(15), we obtain

$$X_{0,s+t} = X_{s,s+t} \circ X_{0,s}, \quad a.s.. \quad (16)$$

Define $X_{s,s+t}^0 := x$ and

$$\begin{aligned}
X_{s,s+t}^n &:= \psi_{s,s+t}\left\{x + \int_s^{s+t} \psi_{s,r}^{-1}(a + \sigma_i b^i)X_{s,r}^{n-1}dr \right. \\
&\quad \left. + \int_s^{s+t} \int_{|u|>\delta} \psi_{s,r}^{-1}\sigma_i X_{s,r}^{n-1}uN_{\kappa^i}(dr, du) \right\}, \quad n \geq 1.
\end{aligned}$$

We claim

$$X_{s,s+t}^n(\omega) = X_{0,t}^n(\theta_s\omega), \quad a.s. \quad (17)$$

for all $n \in \mathbb{N} \cup \{0\}$.

Suppose $X_{s,s+t}^n(\omega) = X_{0,t}^n(\theta_s\omega)$, a.s.. Hence,

$$\begin{aligned}
X_{s,s+t}^{n+1} &= \psi_{s,s+t}\left\{x + \int_s^{s+t} \psi_{s,r}^{-1}(a + \sigma_i b^i)X_{s,r}^n dr \right. \\
&\quad \left. + \int_s^{s+t} \int_{|u|>\delta} \psi_{s,r}^{-1}\sigma_i X_{s,r}^n uN_{\kappa^i}(dr, du) \right\} \\
&= \psi_{0,t}(\theta_s\omega)\left\{x + \int_0^t \psi_{s,s+r}^{-1}(a + \sigma_i b^i)X_{s,s+r}^n dr \right. \\
&\quad \left. + \int_0^t \int_{|u|>\delta} \psi_{s,s+r}^{-1}\sigma_i X_{s,s+r}^n uN_{\kappa^i(\theta_s\omega)}(dr, du) \right\} \\
&= \psi_{0,t}(\theta_s\omega)\left\{x + \int_0^t \psi_{0,r}^{-1}(\theta_s\omega)(a + \sigma_i b^i)X_{0,r}^n(\theta_s\omega)dr \right. \\
&\quad \left. + \int_0^t \int_{|u|>\delta} \psi_{0,r}^{-1}(\theta_s\omega)\sigma_i X_{0,r}^n(\theta_s\omega)uN_{\kappa^i(\theta_s\omega)}(dr, du) \right\} \\
&= X_{0,t}^{n+1}(\theta_s\omega),
\end{aligned}$$

where we have used the fact $\psi_{s,s+t} = \psi_{0,t}(\theta_s\omega)$ ([1, Lemma 6.4.10, p.390]). Taking the limit on two sides of (17) in \mathbb{F} , we have

$$X_{s,s+t}(\omega) = X_{0,t}(\theta_s\omega), \quad a.s.. \quad (18)$$

Combining (16) and (18), we get

$$\begin{aligned} \varphi(t+s, \omega) &= X_{0,s+t}(\omega) = X_{s,s+t} \circ X_{0,s} \\ &= X_{0,t}(X_{0,s}, \theta_s\omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega), \quad a.s.. \end{aligned}$$

Now we prove (iii). Suppose that $\{e_1, e_2, \dots, e_d\}$ is an orthonormal basis of \mathbb{R}^d . So, by the proof of (i) and the fact that

$$\begin{aligned} \|\varphi(t, \omega)\| &= \sup_{x \in B} |\varphi(t, \omega)x| = \sup_{i=1,2,\dots,d} |\varphi(t, \omega)e_i| \\ &= \sup_{i=1,2,\dots,d} |\varphi(t, \omega, e_i)|, \end{aligned}$$

we deduce that $\varphi(t, \omega)$ is strongly measurable. This completes the proof. \square

For $1 \leq i \leq q$, define a one-dimensional two-sided α -stable Lévy motion \hat{L}_t^i , for $t \in \mathbb{R}$,

$$\hat{L}_t^i = \begin{cases} L_t^i, & t \geq 0, \\ \tilde{L}_t^i, & t < 0, \end{cases}$$

where \tilde{L}_t^i is a one-dimensional α -stable Lévy motion of one-sided time $t \leq 0$ and independent of L_t^j for $1 \leq j \leq q$ and \tilde{L}_t^k for $1 \leq k \leq q, k \neq i$. Let us look at the following system

$$\begin{cases} dX_t = aX_t dt + \sigma_i X_t d\hat{L}_t^i, \\ X_0 = x, \end{cases} \quad (19)$$

where the stochastic integral is understood as the forward Itô integral for $t \geq 0$ ([10]) and the backward Itô integral for $t \leq 0$ ([5]). By the similar discussion above for $t \geq 0$, we know that Eq.(19) generates a linear discontinuous cocycle $\varphi(t, \omega)$ for $t \in \mathbb{R}$ and $\omega \in \Omega$. We have the following result about integrability.

Theorem 4.4. $\alpha^\pm \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. Because $\varphi(t, \omega)$ is càdlàg for $t \geq 0$, α^+ is $\mathcal{F}/\mathcal{B}(\mathbb{R}_+)$ -measurable. Furthermore,

$$\begin{aligned} \mathbb{E}\alpha^+ &= \mathbb{E} \left(\sup_{0 \leq t \leq 1} \log^+ \|\varphi(t, \omega)\| \right) = \mathbb{E} \left(\log^+ \sup_{0 \leq t \leq 1} \|\varphi(t, \omega)\| \right) \\ &= \mathbb{E} \left(\log^+ \sup_{0 \leq t \leq 1} \sup_{x \in B} |\varphi(t, \omega)x| \right) = \mathbb{E} \left(\log^+ \sup_{0 \leq t \leq 1} \sup_{x \in B} |X_t(x)| \right) \\ &= \mathbb{E} (\log^+ \|X\|_{\mathbb{F}}), \end{aligned}$$

and by the proof of Theorem 4.3 for $T = 1$

$$\begin{aligned} \|X\|_{\mathbb{F}} &\leq \sum_{n=0}^{\infty} \|X^{n+1} - X^n\|_{\mathbb{F}} + \|x\|_{\mathbb{F}} \\ &\leq \sum_{n=0}^{\infty} \frac{(C_3 \xi_1)^n}{n!} \|X^1 - X^0\|_{\mathbb{F}} + 1 \\ &= \exp\{C_3 \xi_1\} \cdot (C_1 \|x\|_{\mathbb{F}} + C_3 \|x\|_{\mathbb{F}} \xi_1 + \|x\|_{\mathbb{F}}) + 1 \end{aligned}$$

$$= \exp\{C_3\xi_1\} \cdot (C_1 + C_3\xi_1 + 1) + 1.$$

So, to prove $\mathbb{E}\alpha^+ < \infty$, by the inequality $\log(1+x) < x$ for $x > 0$, we only need to consider the integrability of C_1 and $C_3\xi_1$.

By [23, Lemma 2, p.258], $\mathbb{E}C_1 < \infty$. And

$$\begin{aligned} \mathbb{E}(C_3\xi_1) &= (\|a + \sigma_i b^i\| + 1) \mathbb{E} \left[C_1 C_2 \left(1 + \|\sigma_i\| \int_0^1 \int_{|u|>\delta} |u| N_{\kappa^i}(\mathrm{d}s, \mathrm{d}u) \right) \right] \\ &= (\|a + \sigma_i b^i\| + 1) \left(\mathbb{E}[C_1 C_2] + \|\sigma_i\| \mathbb{E} \left[C_1 C_2 \int_0^1 \int_{|u|>\delta} |u| N_{\kappa^i}(\mathrm{d}s, \mathrm{d}u) \right] \right). \end{aligned}$$

By [23, Lemma 2, p. 258] and Hölder inequality, we obtain

$$\mathbb{E}[C_1 C_2] < \infty,$$

and

$$\begin{aligned} &\mathbb{E} \left[C_1 C_2 \int_0^1 \int_{|u|>\delta} |u| N_{\kappa^i}(\mathrm{d}s, \mathrm{d}u) \right] \\ &\leq \left[\mathbb{E} (C_1 C_2)^{(1/\beta)+1} \right]^{\beta/(1+\beta)} \cdot \left[\mathbb{E} \left(\int_0^1 \int_{|u|>\delta} |u| N_{\kappa^i}(\mathrm{d}s, \mathrm{d}u) \right)^{1+\beta} \right]^{1/(1+\beta)} \\ &< \infty, \end{aligned}$$

where $0 < \beta < \alpha - 1$ and the last step is based on [24, proposition 19.5, p. 123] and [24, Corollary 25.8, p. 162].

For α^- , it follows from the cocycle property that

$$\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t \omega).$$

By a similar calculation as that for α^+ , we also conclude that $\mathbb{E}\alpha^- < \infty$. \square

Thus, by Theorem 6.2, multiplicative ergodic theorem for Eq.(10) holds, which generalizes Theorem 5.2 for $r = 0$ in [17].

5. DISCUSSIONS

The noise in the stochastic differential equations does not have to be in terms of α -stable Lévy motions or even other Lévy motions. In fact, we can consider more general linear stochastic differential equations with two-sided semimartingales with stationary increments as in [25, 11]. With an appropriately chosen canonical sample space Ω , consisting of sample paths of a two-sided semimartingale, the multiplicative ergodic theorem, Theorem 6.2 in the Appendix at the end of this paper, also holds. This sets the stage for investigating linear and then nonlinear stochastic differential equations with two-sided semimartingales that have stationary increments. The solution mappings for these stochastic differential equations, after a perfection procedure [25, 11], define random dynamical systems.

6. APPENDIX

For readers' convenience, we state and prove a multiplicative ergodic theorem for discontinuous linear cocycles.

6.1. Flags and related metrics. Let us introduce the definition for a flag of type τ . Let τ be a p -dimensional vector with positive integer components such that $\tau = (d_p, \dots, d_1)$ and $1 \leq d_p < \dots < d_1 = d$. A flag of type τ in \mathbb{R}^d is a sequence of subspaces $F = (V_p, \dots, V_1)$ such that $V_p \subset \dots \subset V_1 = \mathbb{R}^d$ and $\dim V_i = d_i$ for all i . The set for all flags of type τ constitutes the space of flags $F_\tau(d)$. Moreover, $F_\tau(d)$ can be given a structure of a compact C^∞ manifold in a natural way ([9]). And the flag manifold $F_\tau(d)$ can be endowed with a complete metric δ as follows ([6]):

Let U_p be equal to V_p and U_i be the orthogonal complement of V_{i+1} in V_i , $i = p-1, \dots, 1$, so that

$$V_i = U_p \oplus \dots \oplus U_i, \quad i = p, \dots, 1.$$

Define for any $F = (V_p, \dots, V_1), \tilde{F} = (\tilde{V}_p, \dots, \tilde{V}_1) \in F_\tau(d)$

$$\tilde{\rho}(F, \tilde{F}) := \max_{\substack{i \neq j, x \in U_i, y \in \tilde{U}_j; \\ \|x\| = \|y\| = 1}} |\langle x, y \rangle|^{h/|\lambda_i - \lambda_j|},$$

where $\lambda_1, \lambda_2, \dots, \lambda_p$ and h are real numbers which satisfy $\lambda_i \neq \lambda_j$ for $i \neq j$ and

$$h^{-1}|\lambda_i - \lambda_j| \geq d - 1 \text{ for } i \neq j.$$

By Remark 3.4.8 in [2], $\tilde{\rho}(F, \tilde{F})$ can also be written as

$$\tilde{\rho}(F, \tilde{F}) = \max_{i \neq j} \|P_i \tilde{P}_j\|^{h/|\lambda_i - \lambda_j|},$$

where P_i denotes the orthogonal projection onto U_i .

6.2. Multiplicative ergodic theorems for discontinuous linear cocycles. We first recall the following lemma for linear cocycles with discrete time ([2, Theorem 3.4.11(A), p.153]).

Lemma 6.1. (MET for Linear Cocycle with Two-Sided Discrete Time)

Let

$$\varphi(n, \omega) = \begin{cases} A(\theta^{n-1}\omega) \cdots A(\omega), & n \geq 1, \\ I, & n = 0, \\ A^{-1}(\theta^n \omega) \cdots A^{-1}(\theta^{-1}\omega), & n \leq -1, \end{cases}$$

where $A : \Omega \mapsto Gl(d, \mathbb{R})$ ($Gl(d, \mathbb{R})$ denotes the group of $d \times d$ invertible real matrices) is a strongly measurable random invertible matrix and $\theta : \Omega \mapsto \Omega$ is a measurable mapping with $\theta^{-1}\Omega = \Omega$ and $\mathbb{P}\theta^{-1} = \mathbb{P}$. Assume

$$\log^+ \|A(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}) \text{ and } \log^+ \|A^{-1}(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Then there exists an invariant set $\tilde{\Omega}$ of full measure such that for $\omega \in \tilde{\Omega}$

(i) The limit $\lim_{n \rightarrow \infty} (\varphi(n, \omega)^* \varphi(n, \omega))^{1/2n} =: \Phi(\omega) \geq 0$ exists.

(ii) Let $e^{\lambda_{p(\omega)}(\omega)} < \dots < e^{\lambda_1(\omega)}$ be the different eigenvalues of $\Phi(\omega)$ ($\lambda_{p(\omega)} > -\infty$) and let $U_{p(\omega)}(\omega), \dots, U_1(\omega)$ be the corresponding eigenspaces with multiplicities $d_i(\omega) := \dim U_i(\omega)$. Then

$$p(\theta\omega) = p(\omega), \quad \lambda_i(\theta\omega) = \lambda_i(\omega), \quad d_i(\theta\omega) = d_i(\omega),$$

for $i = 1, \dots, p(\omega)$.

(iii) Put $V_{p(\omega)+1}(\omega) := \{0\}$ and for $i = 1, \dots, p(\omega)$

$$V_i(\omega) := U_{p(\omega)}(\omega) \oplus \dots \oplus U_i(\omega),$$

so that

$$V_{p(\omega)}(\omega) \subset \cdots \subset V_i(\omega) \subset \cdots \subset V_1(\omega) = \mathbb{R}^d$$

defines a filtration of \mathbb{R}^d . Then for each $x \in \mathbb{R}^d \setminus \{0\}$ the Lyapunov exponent

$$\lambda(\omega, x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log |\varphi(n, \omega)x|$$

exists and

$$\lambda(\omega, x) = \lambda_i(\omega) \iff x \in V_i(\omega) \setminus V_{i+1}(\omega),$$

equivalently

$$V_i(\omega) = \{x \in \mathbb{R}^d : \lambda(\omega, x) \leq \lambda_i(\omega)\}.$$

(iv) For all $x \in \mathbb{R}^d \setminus \{0\}$

$$\lambda(\theta\omega, A(\omega)x) = \lambda(\omega, x),$$

whence

$$A(\omega)V_i(\omega) = V_i(\theta\omega)$$

for $i = 1, \dots, p(\omega)$.

(v) If $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is ergodic, $p(\omega)$ is a constant on $\tilde{\Omega}$, and $\lambda_i(\omega)$ and $d_i(\omega)$ are two constants on $\{\omega \in \Omega : p(\omega) \geq i\}$, $i = 1, \dots, d$.

(vi) For each $\omega \in \tilde{\Omega}$ there exists a splitting

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_{p(\omega)}(\omega)$$

of \mathbb{R}^d with $\dim E_i(\omega) = d_i(\omega)$ such that for $i \in \{1, \dots, p(\omega)\}$,

(a) if $P_i(\omega) : \mathbb{R}^d \mapsto E_i(\omega)$ is the projection onto $E_i(\omega)$ along $F_i(\omega) := \bigoplus_{j \neq i} E_j(\omega)$, then

$$A(\omega)P_i(\omega) = P_i(\theta\omega)A(\omega),$$

equivalently

$$A(\omega)E_i(\omega) = E_i(\theta\omega),$$

(b) we have

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log |\varphi(n, \omega)x| = \lambda_i(\omega) \iff x \in E_i(\omega) \setminus \{0\},$$

(c) convergence in (b) is uniform with respect to $x \in E_i(\omega) \cap S$ for each fixed ω , where $S = \{x \in \mathbb{R}^d : |x| = 1\}$.

Now we state and prove the following MET for discontinuous linear cocycles.

Theorem 6.2. (MET for Linear Discontinuous Cocycle with Two-Sided Continuous Time)

Let $\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be a linear discontinuous cocycle over the metric DS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$. Let $\varphi(t, \omega) \in Gl(d, \mathbb{R})$. Assume that $\alpha^+ \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\alpha^- \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\alpha^+(\omega) := \sup_{0 \leq t \leq 1} \log^+ \|\varphi(t, \omega)\|, \quad \alpha^-(\omega) := \sup_{0 \leq t \leq 1} \log^+ \|\varphi(t, \omega)^{-1}\|. \quad (20)$$

Then there exists an invariant set $\tilde{\Omega}$ of full measure such that for $\omega \in \tilde{\Omega}$ all statements of Lemma 6.1 hold with n, θ and $A(\omega)$ replaced by t, θ_t and $\varphi(t, \omega)$.

Remark 6.3. In [12, p. 50], Krengel stated a MET for one-sided continuous-time linear cocycle φ that is only measurable in time, such that $\sup_{0 \leq t \leq 1} \log^+ \|\varphi(t, \omega)\|$ and $\sup_{0 \leq t \leq 1} \log^+ \|\varphi(1-t, \theta_t \omega)\|$ are both integrable. Krengel did not prove this MET but mentioned that Oseledet deduced the MET in [19]. It appears to be more general than Theorem 6.2. In fact, firstly, measurability in time does not assure that $\sup_{0 \leq t \leq 1} \log^+ \|\varphi(t, \omega)\|$ and $\sup_{0 \leq t \leq 1} \log^+ \|\varphi(1-t, \theta_t \omega)\|$ are two random variables. Secondly, the integrability of $\sup_{0 \leq t \leq 1} \log^+ \|\varphi(1-t, \theta_t \omega)\|$ is equivalent to the integrability of $\sup_{0 \leq t \leq 1} \log^+ \|\varphi(t, \omega)^{-1}\|$ when $\varphi(t, \omega)$ is invertible. Finally, in [19] Oseledet proved a MET under the conditions that $\sup_{-1 \leq t \leq 1} \log^+ \|\varphi(t, \omega)\|$ and $\sup_{-1 \leq t \leq 1} \log^+ \|\varphi^{-1}(t, \omega)\|$ are both integrable.

For completeness, we provide a proof that is different from Oseledet's one in [19], but follows [3, 14] with the help of Lemma 6.1.

Proof. Step 1. Measurability. Because $\varphi(t, \omega)$ is càdlàg for $t \geq 0$, the random variables α^+ and α^- are \mathcal{F} -measurable.

Step 2. Convergence of flags. (i) For $t \in \mathbb{R}_+$, there exist two orthogonal matrices G_t and O_t such that

$$\varphi(t) = G_t D_t O_t, \quad D_t = \text{diag}(\delta_1(\varphi(t)), \dots, \delta_d(\varphi(t))),$$

where $\delta_i(\varphi(t))$ is the singular value of $\varphi(t)$ and $\delta_1(\varphi(t)) \geq \delta_2(\varphi(t)) \geq \dots \geq \delta_d(\varphi(t)) > 0$. By Proposition 3.2.7 (iii) in [2],

$$\|\wedge^k \varphi(t)\| = \delta_1(\varphi(t)) \cdots \delta_k(\varphi(t)), \quad (21)$$

where $\wedge^k \varphi(t)$ denotes the k -fold exterior power of $\varphi(t)$.

Cocycle property for $\varphi(t)$ and Lemma 3.2.6 (v) in [2] allow us to get

$$\wedge^k \varphi(t, \omega) = (\wedge^k \varphi(t - [t], \theta_{[t]} \omega)) (\wedge^k \varphi([t], \omega))$$

and

$$\begin{aligned} \wedge^k \varphi([t], \omega) &= (\wedge^k \varphi(t - [t], \theta_{[t]} \omega))^{-1} (\wedge^k \varphi(t, \omega)) \\ &= (\wedge^k \varphi(t - [t], \theta_{[t]} \omega)^{-1}) (\wedge^k \varphi(t, \omega)). \end{aligned}$$

Based on Proposition 3.2.7 (iii) in [2], it holds that

$$\begin{aligned} \|\wedge^k \varphi(t, \omega)\| &\leq \|\wedge^k \varphi([t], \omega)\| \|\varphi(t - [t], \theta_{[t]} \omega)\|^k \\ &\leq \|\wedge^k \varphi([t], \omega)\| \left(\sup_{0 \leq s \leq 1} \|\varphi(s, \theta_{[t]} \omega)\| \right)^k \end{aligned}$$

and

$$\begin{aligned} \|\wedge^k \varphi([t], \omega)\| &\leq \|\varphi(t - [t], \theta_{[t]} \omega)^{-1}\|^k \|\wedge^k \varphi(t, \omega)\| \\ &\leq \left(\sup_{0 \leq s \leq 1} \|\varphi(s, \theta_{[t]} \omega)^{-1}\| \right)^k \|\wedge^k \varphi(t, \omega)\|. \end{aligned}$$

Thus,

$$\frac{\log \|\wedge^k \varphi([t], \omega)\|}{t} - \frac{k\alpha^-(\theta_{[t]} \omega)}{t} \leq \frac{\log \|\wedge^k \varphi(t, \omega)\|}{t}$$

$$\leq \frac{\log \|\wedge^k \varphi([t], \omega)\|}{t} + \frac{k\alpha^+(\theta_{[t]}\omega)}{t},$$

where we have used the following two inequalities:

$$\begin{aligned} \log \sup_{0 \leq s \leq 1} \|\varphi(s, \theta_{[t]}\omega)\| &\leq \sup_{0 \leq s \leq 1} \log^+ \|\varphi(s, \theta_{[t]}\omega)\|, \\ \log \sup_{0 \leq s \leq 1} \|\varphi(s, \theta_{[t]}\omega)^{-1}\| &\leq \sup_{0 \leq s \leq 1} \log^+ \|\varphi(s, \theta_{[t]}\omega)^{-1}\|. \end{aligned}$$

By (20), we obtain that

$$\lim_{n \rightarrow \infty} \frac{\log \|\wedge^k \varphi(n, \omega)\|}{n} = \lim_{t \rightarrow \infty} \frac{\log \|\wedge^k \varphi(t, \omega)\|}{t}.$$

So, by Theorem 3.3.3(B) in [2] for $A(\omega) = \varphi(1, \omega)$ and $\theta = \theta_1$, there exist a forward invariant set $\Omega_1 \in \mathcal{F}$ of full measure ($\Omega_1 \subset \theta_1^{-1}\Omega_1$ and $\mathbb{P}(\Omega_1) = 1$) and measurable functions $\gamma^{(k)} : \Omega \rightarrow \mathbb{R}$, with $(\gamma^{(k)})^+ \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, such that

$$\lim_{t \rightarrow \infty} \frac{\log \|\wedge^k \varphi(t, \omega)\|}{t} = \gamma^{(k)}, \quad a.s.. \quad (22)$$

Combining (22) and (21), we have

$$\lim_{t \rightarrow \infty} D_t^{1/t} = \text{diag}(e^{\Lambda_1}, \dots, e^{\Lambda_d}),$$

where $\gamma^{(k)} = \Lambda_1 + \dots + \Lambda_k$.

Denote by $\lambda_1 > \dots > \lambda_p$ the distinct numbers among the Λ_i . Let d_i be the multiplicity of λ_i for $i = 1, \dots, p$. Put

$$\Delta_i := \lambda_i - \lambda_{i+1}, i = 1, \dots, p-1, \quad \Delta := \min_{i=1, \dots, p-1} \Delta_i > 0.$$

Let $U_i(t)$ be spanned by the group Σ_i of those eigenvectors of $(\varphi^*(t)\varphi(t))^{1/2t} = O_t^* D_t^{1/t} O_t$ corresponding to eigenvalues $\delta_{k(i)}(\varphi(t))^{1/t}$, where $\lim_{t \rightarrow \infty} \delta_{k(i)}(\varphi(t))^{1/t} = e^{\lambda_i}$, and

$$V_i(t) := U_p(t) \oplus \dots \oplus U_i(t), \quad i = 1, \dots, p.$$

The sequence of subspaces of \mathbb{R}^d given by

$$F(t) = (V_p(t), \dots, V_i(t), \dots, V_1(t))$$

forms a flag of type

$$\tau = (d_p, d_p + d_{p-1}, \dots, d_p + \dots + d_1 = d).$$

Denote $h = \frac{\Delta}{d-1}$. So, by Section 6.1, the distance between $F(t)$ and $F([t])$, in $F_\tau(d)$, is given by

$$\tilde{\rho}(F(t), F([t])) = \max_{\substack{i \neq j \\ i, j=1, \dots, p}} \|P_i(t)P_j([t])\|^{h/|\lambda_i - \lambda_j|},$$

where $P_i(t)$ denotes the orthogonal projection onto $U_i(t)$.

(ii) Next, we calculate $\tilde{\rho}(F(t), F([t]))$. If $i > j$, $\lambda_i < \lambda_j$. Take a unit vector $x \in U_i([t])$ and $y = P_j(t)x \in U_j(t)$. Thus,

$$\begin{aligned} |\varphi(t, \omega)x| &= |\varphi(t - [t], \theta_{[t]}\omega)\varphi([t], \omega)x| \leq \|\varphi(t - [t], \theta_{[t]}\omega)\| \|\varphi([t], \omega)x\| \\ &\leq \|\varphi(t - [t], \theta_{[t]}\omega)\| \bar{\delta}_i(\varphi([t])), \end{aligned}$$

and

$$\begin{aligned} |\varphi(t, \omega)x|^2 &= |\varphi(t, \omega)y|^2 + |\varphi(t, \omega)(x - y)|^2 \\ &\geq |\varphi(t, \omega)y|^2 \geq \underline{\delta}_j(\varphi(t))^2 |y|^2, \end{aligned}$$

where

$$\bar{\delta}_i(\varphi([t])) = \sup_{\Sigma_i} \delta_{k(i)}(\varphi([t])), \quad \underline{\delta}_j(\varphi(t)) = \inf_{\Sigma_j} \delta_{k(j)}(\varphi(t)),$$

with

$$\lambda_i = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \bar{\delta}_i(\varphi([t])), \quad \lambda_j = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \underline{\delta}_j(\varphi(t)).$$

Therefore,

$$|y| = |P_j(t)P_i([t])x| \leq \frac{|\varphi(t, \omega)x|}{\underline{\delta}_j(\varphi(t))} \leq \|\varphi(t - [t], \theta_{[t]}\omega)\| \frac{\bar{\delta}_i(\varphi([t]))}{\underline{\delta}_j(\varphi(t))},$$

and

$$\begin{aligned} \|P_j(t)P_i([t])\| &\leq \|\varphi(t - [t], \theta_{[t]}\omega)\| \frac{\bar{\delta}_i(\varphi([t]))}{\underline{\delta}_j(\varphi(t))} \\ &\leq \sup_{0 \leq s \leq 1} \|\varphi(s, \theta_{[t]}\omega)\| \frac{\bar{\delta}_i(\varphi([t]))}{\underline{\delta}_j(\varphi(t))}. \end{aligned}$$

Moreover,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \|P_j(t)P_i([t])\| \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log \sup_{0 \leq s \leq 1} \|\varphi(s, \theta_{[t]}\omega)\| - |\lambda_i - \lambda_j|.$$

Since $\log \sup_{0 \leq s \leq 1} \|\varphi(s, \omega)\| \leq \sup_{0 \leq s \leq 1} \log^+ \|\varphi(s, \omega)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, we deduce that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \sup_{0 \leq s \leq 1} \|\varphi(s, \theta_{[t]}\omega)\| \leq 0.$$

Thus,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \|P_j(t)P_i([t])\| \leq -|\lambda_i - \lambda_j|. \quad (23)$$

If $i < j$, $\lambda_i > \lambda_j$. By the same deduction as above, we obtain

$$\|P_i([t])P_j(t)\| \leq \|\varphi(t - [t], \theta_{[t]}\omega)^{-1}\| \frac{\bar{\delta}_j(\varphi(t))}{\underline{\delta}_i(\varphi([t]))}$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \|P_j(t)P_i([t])\| \leq -|\lambda_i - \lambda_j|. \quad (24)$$

Combining (24) and (23), we get

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{\rho}(F(t), F([t])) \leq -h. \quad (25)$$

(iii) By Lemma 3.4.9 in [2], there exists a flag $F = (V_p, \dots, V_i, \dots, V_1)$ of type τ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \tilde{\rho}(F(n), F) \leq -h. \quad (26)$$

By (25) and (26), we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \tilde{\rho}(F(t), F) &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left(\tilde{\rho}(F(t), F([t])) + \tilde{\rho}(F([t]), F) \right) \\ &\leq -h. \end{aligned}$$

Step 3. Lyapunov exponent. If $t = n + s$ with $s \in (0, 1), n \in \mathbb{N}$, then

$$\|\varphi(s, \theta_n \omega)\| \|\varphi(n, \omega)x\| \geq |\varphi(t, \omega)x| \geq \|\varphi(s, \theta_n \omega)^{-1}\|^{-1} |\varphi(n, \omega)x|,$$

and therefore

$$\alpha^+(\theta_n \omega) + \log |\varphi(n, \omega)x| \geq \log |\varphi(t, \omega)x| \geq \log |\varphi(n, \omega)x| - \alpha^-(\theta_n \omega).$$

Since $\lim_{n \rightarrow \infty} n^{-1} \alpha^+(\theta_n \omega) = \lim_{n \rightarrow \infty} n^{-1} \alpha^-(\theta_n \omega) = 0$ with probability 1, one has

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log |\varphi(t, \omega)x| = \lim_{n \rightarrow \infty} \frac{1}{n} \log |\varphi(n, \omega)x|.$$

By Lemma 6.1 (iii), the statement in (iii) holds.

Step 4. Invariancy. For $x \in \mathbb{R}^d \setminus \{0\}$,

$$\begin{aligned} \lambda(\theta_t \omega, \varphi(t, \omega)x) &= \limsup_{s \rightarrow \infty} \frac{1}{s} \log |\varphi(s, \theta_t \omega) \varphi(t, \omega)x| \\ &= \limsup_{s \rightarrow \infty} \frac{1}{s} \log |\varphi(s + t, \omega)x| \\ &= \limsup_{s \rightarrow \infty} \frac{1}{s + t} \log |\varphi(s + t, \omega)x| \cdot \frac{s + t}{s} \\ &= \lambda(\omega, x). \end{aligned}$$

Step 5. The flag for negative time. For $t \in \mathbb{R}_-$, cocycle property for $\varphi(t, \omega)$ infers that

$$\varphi(t, \omega) = \varphi(-t, \theta_t \omega)^{-1}.$$

Let $\delta_1(\varphi(-t, \theta_t \omega)) \geq \delta_2(\varphi(-t, \theta_t \omega)) \geq \dots \geq \delta_d(\varphi(-t, \theta_t \omega)) > 0$ be singular values of $\varphi(-t, \theta_t \omega)$, and then these singular values $\delta_1(\varphi(t)) \geq \delta_2(\varphi(t)) \geq \dots \geq \delta_d(\varphi(t))$ of $\varphi(t)$ satisfy

$$\delta_k(\varphi(t)) = \delta_{d+1-k}(\varphi(-t, \theta_t \omega))^{-1},$$

for $k = 1, 2, \dots, d$. By the same deduction as that in Step 2, we get that

$$\lim_{n \rightarrow -\infty} \frac{\log \|\wedge^k \varphi(n, \omega)\|}{|n|} = \lim_{t \rightarrow -\infty} \frac{\log \|\wedge^k \varphi(t, \omega)\|}{|t|}.$$

So, by Theorem 3.3.10(A) in [2] for $A^{-1}(\theta^{-1} \omega) = \varphi(-1, \omega)$ and $\theta^{-1} = \theta_{-1}$, on a invariant set $\Omega_2 \in \mathcal{F}$ of full measure

$$\lim_{n \rightarrow -\infty} \frac{\log \|\wedge^k \varphi(n, \omega)\|}{|n|} = \gamma^{(d-k)} - \gamma^{(d)}, \quad a.s.. \quad (27)$$

Combining (27) and (21), we have

$$\lim_{t \rightarrow -\infty} D_t^{1/t} = \text{diag}(e^{\Lambda_1^-}, \dots, e^{\Lambda_d^-}),$$

where $\Lambda_k^- = -\Lambda_{d+1-k}$.

Denote by $\lambda_1^- > \cdots > \lambda_p^-$ the distinct numbers among Λ_i^- . Let d_i^- be the multiplicity of λ_i^- for $i = 1, \dots, p$. Then

$$\lambda_k^- = -\lambda_{p+1-k}, \quad d_k^- = d_{p+1-k}.$$

By the same deduction as in Step 2, we get a flag $F^- = (V_p^-, V_{p-1}^-, \dots, V_1^-)$ of type $\tau^- = (d_p^-, d_p^- + d_{p-1}^-, \dots, d_p^- + \cdots + d_1^- = d)$.

Step 6. Oseledets spaces. Let

$$E_i = V_i \cap V_{p+1-i}^-, \quad i = 1, 2, \dots, p.$$

So, by the proof of Theorem 3.4.11(A) in [2], E_1, E_2, \dots, E_p , which form a splitting of \mathbb{R}^d , are Oseledets spaces.

In the following, we examine the properties of Oseledets spaces E_i .

(i) For $t \geq 0$ by Step 4, and for $t \leq 0$, similar to that in Step 4, we obtain

$$\begin{aligned} \varphi(t, \omega)E_i &= \varphi(t, \omega)V_i \cap \varphi(t, \omega)V_{p+1-i}^- \\ &= V_i(\theta_t \omega) \cap V_{p(\theta_t \omega)+1-i}^-(\theta_t \omega) \\ &= E_i(\theta_t \omega). \end{aligned}$$

(ii) For $t \geq 0$ by Step 3, and for $t \leq 0$ similar to that in Step 3, it holds that

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log |\varphi(t, \omega)x| = \lim_{n \rightarrow \pm\infty} \frac{1}{n} \log |\varphi(n, \omega)x|.$$

Thus, by Lemma 6.1(vi)(b), we have

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log |\varphi(t, \omega)x| = \lambda_i(\omega) \iff x \in E_i(\omega) \setminus \{0\}.$$

The proof is thus completed. \square

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